The Monopole Dominance in QCD

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Abstract

A projection (gauge) independent formulation of the monopole dominance, discovered in lattice QCD for the maximal abelian projection, is given. A new dynamical abelian projection of continuum QCD, which does not rely on any explicit gauge condition imposed on gauge fields, is proposed. Under the assumption that the results of numerical simulations hold in the continuum limit, the monopole dominance is proved for the dynamical abelian projection. The latter enables us to develop an effective scalar field theory for dominant (monopole) configurations of gauge fields. The approach is manifestly gauge and Lorentz invariant.

1. The Coulomb problem in QCD

To calculate an interaction energy $V(R)$ of two static color sources separated by a distance R in non-abelian gauge theory, one has to analyze the behavior of the Wilson loop expectation value in the large Euclidean time limit

$$\langle W_R\rangle = \langle 1\rangle^{-1}\int \mathcal{D}A_\mu e^{-S_{YM}}\mathrm{tr}\, Pe^{ig\oint_{C_R} dx_\mu A_\mu} \to e^{-TV(R)}\ , \tag{1}$$

as $T\to\infty$. The integral (1) is taken over configurations satisfying periodic boundary conditions $A_\mu(\mathbf{x},0)=A_\mu(\mathbf{x},T)$, and S_{YM} is the Euclidean Yang-Mills action for a finite Euclidean time T; the exponential $e^{-S_{YM}}$ serves as the probability distribution in (1).

In electrodynamics fluctuations of electromagnetic fields are gaussian and described by an infinite set of harmonic oscillators driven by an external force. Doing the integral (1) and taking the limit, one finds that $V(R)\sim R^{-1}$, i.e. the Coulomb law. In the case of Yang-Mills theory, the integral is not gaussian. The theory describes an infinite set of coupled anharmonic oscillators driven by an external force. Therefore, some approximate method to evaluate the integral (1) should be developed. It is well known that the perturbation theory for the integral (1) leads to the Coulomb law at short distances, which reflects the asymptotic freedom in QCD, whereas for large distances, the perturbation expansion breaks down and fails to reproduce the expected linearly raising potential

$$V(R) = V_0 + \sigma R + O(R^{-1}) \tag{2}$$

that provides the quark confinement.

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A difficulty to develop an approximate approach is due to the lack of understanding what configurations of gauge fields give a main contribution to the integral (1) and, hence, are responsible for the QCD confinement mechanism. A priori there is no clue for how these configuration may look like. Fortunately, recent results of numerical simulations of the lattice QCD show that such configurations do exist, and they look like Dirac magnetic monopoles when taken in a specific gauge called the maximal abelian gauge. Therefore, it is natural to attempt to construct an effective theory for their dynamics. The present letter is devoted to this problem.

2. Monopoles in abelian projections

The idea to associate the dominant configurations with magnetic monopoles attracts much attention because it gives a transparent physical picture for the confinement mechanism where the QCD vacuum is assumed to be a dual superconductor formed by condensed monopole-antimonopole pairs [1]. Magnetic monopoles, as classical soliton-like excitations, are hard to introduce in the non-abelian gauge theory with a simply connected compact group (like, e.q. SU(3)) without a Higgs mechanism, i.e. when the gauge symmetry remains unbroken. A way out was suggested by 't Hooft who proposed a partial gauge fixing, $\chi(A) = 0$, to restrict the gauge group G to its maximal abelian subgroup G_H [2]. When lifted onto the gauge fixing surface $\chi(A) = 0$ by a suitable gauge transformation, gauge potentials appear to have topological singularities occurring through the lift (projection). For the unitary groups $G = SU(n)$ these singularities can be viewed as magnetic monopoles relative to the unbroken electromagnetic group $G_H = (U(1))^{n-1}$.

To see how the topological singularities occur in an abelian projection, let us pick up a local gluon operator $\Gamma(A)$ that transforms according to the adjoint representation

$$\begin{aligned} \Gamma(A) &\;\rightarrow\; \Gamma(A^\Omega) = \Omega\Gamma(A)\Omega^\dagger \ , &(3)\\ A_\mu &\;\rightarrow\; A_\mu^\Omega = \Omega A_\mu \Omega^\dagger + ig^{-1}\Omega\partial_\mu\Omega^\dagger \ . &(4)\end{aligned}$$

The gauge condition breaking G to G_H is

$$\chi(A) = \Gamma^{off}(A) = 0 \ , \tag{5}$$

where "off" stands for the off-diagonal elements of Γ in a matrix representation. For example, one can take $\Gamma(A) = F_{12}(A)$. Clearly, the condition (5) is invariant relative to the abelian transformations from G_H and, hence, does not break G_H. Given a configuration A_μ, consider a gauge group element Ω_Γ such that the element

$$\Gamma_H = \Omega_\Gamma\Gamma(A)\Omega_\Gamma^\dagger \tag{6}$$

belongs to the Cartan subalgebra (i.e. diagonal in the matrix representation). Then the potential

$$A_\mu^{\Omega_\Gamma} = \Omega_\Gamma A_\mu \Omega_\Gamma^\dagger + ig^{-1}\Omega_\Gamma\partial_\mu\Omega_\Gamma^\dagger \tag{7}$$

satisfies the gauge condition (5). In other words, the gauge transformation (7) lifts the configuration A_μ onto the gauge fixing surface (5); the function $\chi(A^{\Omega_\Gamma})$ identically vanishes for any configuration A_μ and the corresponding group element Ω_Γ.

Doing this abelian projection for all configurations A_μ, we arrive at the abelian gauge theory where diagonal (Cartan) components, denoted below as C^Γ_μ, of the projected potentials (7) play the role of the Maxwell fields, while the off-diagonal components are charged fields. We set

$$A^{\Omega_\Gamma}_\mu = C^\Gamma_\mu + W^\Gamma_\mu \ . \tag{8}$$

Under residual gauge transformations, they behave as

$$C^\Gamma_\mu \to C^\Gamma_\mu + \partial_\mu\omega \ ; \quad W^\Gamma_\mu \to e^{ig\omega}W^\Gamma_\mu e^{-ig\omega} \ , \qquad e^{-ig\omega} \in G_H \ . \tag{9}$$

The abelian theory thus obtained is not the usual one. Not for every configuration A_μ, the projective group element $\Omega_\Gamma(A)$ determined by (6) is a well defined function in spacetime. Let us take, for example, the group SU(2), then $\Gamma = \Gamma_a\tau_a$, τ_a are the Pauli matrices, is a 2×2 traceless hermitian matrix with elements being functions of spacetime. The group element $\Omega_\Gamma = \Omega_\Gamma(x)$ is well defined everywhere in spacetime, but the points where $\Gamma_a(x)$ vanish. Three equations $\Gamma_a(x) = 0$ for four spacetime coordinates determine a worldline $x_\mu = x_\mu(\tau)$ (or a set of worldlines). Through the projection transformation (7) the singularities of Ω_Γ are transferred to the potentials $A^{\Omega_\Gamma}_\mu$. Singularities of the Maxwell fields appear to be Dirac magnetic monopoles so that the equations $\Gamma_a(x) = 0$ determine their worldlines.

In the SU(2) case, the Maxwell field has only one component

$$C^\Gamma_\mu = \tfrac{1}{2}\mathrm{tr}\,\tau_3\left(\Omega_\Gamma A_\mu \Omega^\dagger_\Gamma + ig^{-1}\Omega_\Gamma\partial_\mu\Omega^\dagger_\Gamma\right) \ . \tag{10}$$

Let a configuration A_μ be such that $\Gamma_a(\mathbf{x}_m) = 0$ at some isolated spatial point $\mathbf{x} = \mathbf{x}_m$ (time is fixed). Imagine a sphere Σ_m centered at $\mathbf{x}_m$ and a closed contour L_s on it. The magnetic field flux through L_s is

$$\Phi_{L_s} = \oint_{L_s} (\mathbf{C}^\Gamma, d\mathbf{l}) \ . \tag{11}$$

Consider the limit when the contour L_s shrinks to some point $\mathbf{x}_s$ on the sphere. If $\mathbf{C}^\Gamma$ is regular everywhere on the sphere, the flux vanishes, $\Phi_{L_s\to 0} = 0$, for any $\mathbf{x}_s$. Since Ω_Γ is regular on the sphere, the first term in (10) is regular too and gives no finite contribution to the flux (11) in the limit $L_s \to 0$. In contrast, the second term can contain a total derivative of angular (cyclic) variables parametrizing the group element Ω_Γ in the reference frame originated at $\mathbf{x}_m$. If this happens, the flux $\Phi_{L_s\to 0}$ does not vanish at some points $\mathbf{x}_s$ on the sphere Σ_m. The latter means that the magnetic field $\mathbf{B}^\Gamma = \boldsymbol{\nabla}\times\mathbf{C}^\Gamma$ has a string-like singularity which carries a finite magnetic flux and, therefore, can be associated with the Dirac string of a magnetic monopole that pierces L_s. An explicit example of a monopole-carrying group element Ω_Γ is given in section 4.

The origin of this topological singularity can be understood in the following way. On the sphere Σ_m we define a compact field $e_\Gamma = \Omega^\dagger_\Gamma\tau_3\Omega_\Gamma$. Clearly, it is regular on the sphere and, hence, determines a map of the sphere Σ_m onto the sphere $\mathrm{tr}\, e^2_\Gamma = 2$ in the isotopic

space. This map is characterized by the Poincare-Hopf index, being the number of times one sphere is wrapped about the other,

$$q_\Gamma(x_m) = (32\pi i)^{-1}\oint_{\Sigma_m} d\sigma_j \epsilon_{jkn} \mathrm{tr}\,(e_\Gamma[\partial_k e_\Gamma, \partial_n e_\Gamma])\ . \tag{12}$$

Making use of the Stokes theorem we write (11) as a surface integral

$$\Phi_{L_s} = -\int_{\Sigma_m^s} d\sigma_j \epsilon_{jkn}\partial_k C_n^\Gamma = -\int_{\Sigma_m^s} (d\boldsymbol{\sigma}, \mathbf{B}^\Gamma)\,, \tag{13}$$

where Σ_m^s is the sphere Σ_m with a small hole cut out by the contour L_s. In the limit $L_s \to 0$, the surface Σ_m^s turns into Σ_m without the point $\mathbf{x}_s$. Substituting the singular part of (10),

$$\bar{C}_n^\Gamma = -(2ig)^{-1}\mathrm{tr}\,(\tau_3\Omega_\Gamma\partial_n\Omega_\Gamma^\dagger) = (2ig)^{-1}\mathrm{tr}\,(e_\Gamma\Omega_\Gamma^\dagger\partial_n\Omega_\Gamma)\ , \tag{14}$$

in (13), we find after simple algebraic transformations

$$\begin{aligned} \Phi_{L_s\to 0} &= \tfrac{1}{16ig}\oint_{\Sigma_m^s} d\sigma_i\varepsilon_{ijk}\mathrm{tr}\ (e_\Gamma[\partial_j e_\Gamma, \partial_k e_\Gamma]) + \tfrac{1}{2ig}\oint_{\Sigma_m^s} d\sigma_i\varepsilon_{ijk}\mathrm{tr}\ \left(\tau_3\Omega_\Gamma\partial_j\partial_k\Omega_\Gamma^\dagger\right) & (15)\\ &= 2\pi g^{-1}q_\Gamma(x_m) \equiv 4\pi g_m\ . & (16)\end{aligned}$$

At the singular point $\mathbf{x}_s$, we have $[\partial_j, \partial_k]\Omega_\Gamma \neq 0$ so that the last term in (15) does not contribute (the Dirac string pierces the sphere Σ at this point). Therefore the magnetic charge of the Dirac monopole located at $\mathbf{x} = \mathbf{x}_m$ is equal to $g_m = -q_\Gamma/2g$.

Thus, after the abelian projection we obtain electrodynamics with Dirac magnetic monopoles that are topological singularities in the abelian part of the projected gauge fields.

The construction can be extended to any abelian projection determined by a gauge condition $\chi(A) = 0$ that breaks the gauge group to its maximal abelian subgroup. For every configuration A_μ we determine a gauge group element Ω_χ by the equation

$$\chi(A^{\Omega_\chi}) = 0\ . \tag{17}$$

If λ^α, $\alpha = 1, 2, ..., \mathrm{rank}\, G$, is an orthonormal basis in the Cartan subalgebra, then the scalar fields, that characterize topological singularities in the Maxwell fields $C_\mu^\chi = \lambda^\alpha \mathrm{tr}\,(\lambda^\alpha A_\mu^{\Omega_\chi})$, are defined as $e_\chi^\alpha = \Omega_\chi^\dagger\lambda^\alpha\Omega_\chi$. They determine a map of the sphere Σ_m to the coset space G/G_H. The homotopy group of this mapping is $\Pi_2(G/G_H) = \Pi_1(G_H) = \mathbb{Z}^{\mathrm{rank}\ G}$ (here $G = SU(n)$). Therefore the monopole distribution and their charges are still determined by rank G relations (12). Every configuration A_μ is thus associated with a certain set integer-valued functions $q_\chi^\alpha(x)$.

So, the abelian projection χ divides the space $[A]$ of all configurations A_μ into two subspaces characterized by $q_\chi \neq 0$ and by $q_\chi = 0$. Upon the abelian projection χ, the abelian components of configurations from the former ("monopole") subspace turn into the Dirac magnetic monopole potentials with the magnetic charge distribution q_χ, while configurations from the other subspace give rise to no monopole after the projection. We

remark that, following a tradition in lattice gauge theories we call the subspace with $q_\chi \neq 0$ the "monopole" subspace, but we put the quotation marks in order to emphasize the fact that *before* the χ-projection, configurations from this subspace are no Dirac monopoles. The above two subspaces in $[A]$ are defined up to *regular* gauge transformations. Note that the magnetic charge distribution (12) is invariant under transformations

$$\Omega_\chi \to \Omega_0 \Omega_\chi \ , \tag{18}$$

where Ω_0 is regular in spacetime. It is also important to realize that the above definition of the "monopole" subspace is projection (or *gauge*) dependent. Different choices of χ lead, in general, to different "monopole" subspaces in $[A]$. In next section we give a gauge invariant (projection independent) definition of the "monopole" subspace *selected* by abelian projections.

3. Gauge invariance and the monopole dominance

In lattice gauge theory, the integral (1) can be done numerically. As expected, the potential extracted from the lattice simulations has the form (2), and the coefficient σ (called the QCD string tension) is known. Now one can perform the abelian projection on the lattice [3], i.e. the above described procedure of dividing the space $[A]$ of all configurations A_μ into the "monopole" and "non-monopole" subspaces can be performed numerically for a given projection χ. To understand how big a contribution of the monopole configurations to the QCD string tension, one has just to restrict in (1) the sum over configurations by the "monopole" subspace and evaluate the string tension. It turns out that for the so called maximal abelian projection [2, 3]

$$\chi_{ma}(A) = \partial_\mu A_\mu^{off} + ig[A_\mu^H, A_\mu^{off}] = 0 \ , \tag{19}$$

where $A_\mu = A_\mu^H + A_\mu^{off}$, and A_μ^H are Cartan (diagonal) components of A_μ, the difference between an exact string tension and the string tension evaluated on the monopole configurations is only eight per cent [4] (see also recent simulations [5]). This phenomenon is known as the monopole dominance.

Based on the fact of the monopole dominance, one can conjecture that the confinement is due to the condensation of monopole-antimonopole pairs. However, such a conclusion is not straightforward. Recall that this mechanism is theoretically well understood only in gauge theories with the spontaneous gauge symmetry breaking [7], where monopoles are solitons of classical equations of motion. In QCD the color local symmetry remains unbroken and, therefore, the monopole-antimonopole condensation in the abelian projection, if it occurs, is not due to the usual Higgs mechanism.

It should be noted that the results of numerical simulations suggest only that in the space of all configurations $[A]$ there is a relatively "small" subset $[\bar{A}]$ that gives a dominant contribution to the expectation value of the Wilson loop. It seems also that a major part of the dominant subset $[\bar{A}]$ can be *selected* via the maximal abelian projection (19). A natural question arises from the results of numerical simulations. Does there

exist any gauge invariant formulation of the monopole dominance? In fact, the monopole dominance has been observed only in the maximal abelian projection and it is absent in other projections studied on the lattice, although this does not exclude the existence of some other abelian projections that could exhibit the monopole dominance [8]. Below we answer this question.

Consider an abelian projection generated by a gauge condition $\chi(A) = 0$. Topological singularities can only occur in the projected configuration $A_\mu^{\Omega_\chi}$ through singularities in Ω_χ determined by equation (17). Were the group elements Ω_χ regular for all configurations A_μ, there would exist a global gauge condition (in the mathematical language, a global cross-section in the space of all connections $[A]$) determined by $\chi(A) = 0$ and $\partial_\mu C_\mu = 0$, where C_μ is the abelian component of A_μ, which is impossible according to the Singer theorem [9]. Note that the gauge condition $\partial_\mu C_\mu = 0$ breaks the residual abelian gauge group. Since Singer's arguments do not apply to abelian gauge theories (the condition $\partial_\mu C_\mu = 0$ exists globally), we conclude that any abelian projection exhibits topological singularities. In some sense, it is not an accident that configurations having topological defects *after* the projection turn out to be dominant. These configurations are "pure non-abelian" (homotopicaly nontrivial) and contain information about the gauge orbit space topology in Yang-Mills theory.

Topological defects can generally occur in both the abelian C_μ^χ and non-abelian W_μ^χ components of projected configurations, depending on the choice of χ. To those occurring in C_μ^χ we shall refer as monopoles. One can construct an abelian projection where no monopole singularities are possible, i.e. all singularities occur in W_μ^χ. It was conjectured that such singularities may also be associated with dominant configurations [10], although it has not been verified numerically yet. In what follows we consider only monopole singularities. It is worth emphasizing again that the dominant configurations do not depend on any gauge choice. The gauge fixing is used only to *select* the dominant configurations via topological defects occurring upon the abelian projection, and, by now, there is no theoretical explanation of why such a selection works in certain abelian projections (see section 5). In general, there might exist some other ways to characterize them by topological quantities different from the monopole distribution (12) (for instance, in [6] a correlation between the instanton and monopole topological numbers has been observed).

To specify the dominant "monopole" subset associated with a projection χ, we invoke another remarkable result of numerical simulations known as the abelian dominance [11]. It states that the so called abelian string tension, the one that is calculated by averaging the Wilson loop over only the abelian components C_μ^χ of the projected configurations $A_\mu^{\Omega_\chi}$, differs from the full QCD string tension by approximately eight per cent. Therefore the off-diagonal components W_μ^χ of the projected configurations are irrelevant for the formation of the flux tube between static sources in QCD. This has been again verified in the maximal abelian projection and may not be the case in another projection. The dominant subset amongst the projected configurations used to obtain the abelian string tension is wider than the set of monopole configurations in C_μ^χ because it involves fluctuations of monopole-free Maxwell (or photon) fields. The monopole dominance implies that contributions of photons to the string tension are negligible as compared with that of Dirac monopoles [4].

In the lattice simulations, the off-diagonal components W_μ^χ of the projected configurations are set therefore to be zero. It is not acceptable in the continuum case because the magnetic field energy of Dirac monopoles $\bar{C}_\mu^\chi$ (14) is infinite as the monopoles are pointlike particles. Note that in the lattice gauge theory the lattice spacing plays the role of the regularization parameter at short distances. Since the string tension is not sensitive to a particular form of W_μ^χ, we can choose them to provide a core (a finite size) for the Dirac monopoles. An explicit construction of the core functions can be found in [12]. With every monopole distribution $q_\chi(x) \neq 0$ obtained in the abelian projection χ we associate the Dirac monopole configuration $\bar{C}_\mu^\chi$ (14) and a core function $W_\mu^\chi = \bar{W}_\mu^\chi(\bar{C})$ corresponding to it (the Wu-Yang monopole with a core). In the projected theory, the dominant configurations are $\bar{C}_\mu^\chi + \bar{W}_\mu^\chi$. The color magnetic energy of these configurations is finite, so is the action at finite temperature.

We define a subspace $[{}^\chi\bar{A}]$ in $[A]$ by the condition that any configuration ${}^\chi\bar{A}_\mu$ from $[{}^\chi\bar{A}]$ becomes a Dirac monopole with a core when projected on the gauge fixing surface $\chi = 0$, that is,

$$
{}^\chi\bar{A}_\mu^{\Omega_\chi} = \bar{W}_\mu^\chi(\bar{C}^\chi) + \bar{C}_\mu^\chi \ . \tag{20}
$$

The set $[q_\chi]$ of all possible monopole distributions q_χ occurring in an abelian projection χ is isomorphic to the subspace $[{}^\chi\bar{A}]$ modulo *regular* gauge transformations. Note that due to the invariance of $q_\chi(x)$ under the transformations (18), configurations ${}^\chi\bar{A}_\mu$ and ${}^\chi\bar{A}_\mu^{\Omega_0}$ yield the same $q_\chi(x)$ upon the projection.

Consider a subspace $[\bar{A}]$ in $[A]$ that is a union of the spaces $[{}^\chi\bar{A}]$ for all possible projections,

$$
[\bar{A}] = \cup_\chi [{}^\chi\bar{A}] \ . \tag{21}
$$

By construction, this subspace is *projection independent*. For any configuration $\bar{A}_\mu$ from $[\bar{A}]$ there exists an abelian projection gauge χ such that $\bar{A}_\mu^{\Omega_\chi}$ is the vector potential of the form (20), i.e. it describes a set of Dirac monopoles with cores. The subspace (21) modulo regular gauge transformations is isomorphic to a set of all monopole distributions $[\bar{q}]$ that can be obtained in all possible abelian projections of QCD

$$
[\bar{q}] = \cup_\chi [q_\chi] \sim [\bar{A}]/[\Omega_0] \ . \tag{22}
$$

If the monopole dominance occurs, at least, in one abelian projection, then the gauge invariant subspace $[\bar{A}]/[\Omega_0]$ dominates in the path integral (1) because it is larger than any of the "monopole" subspaces $[{}^\chi\bar{A}]$. At this point we assume that the monopole dominance discovered for the maximal abelian projection in lattice QCD survives the continuum limit and, therefore, the integral (1) is dominated by configurations from $[\bar{A}]$. The dependence of the monopole dominance on the abelian projection choice is easily understood. The projection (or gauge) dependent "monopole" space $[{}^\chi\bar{A}]$ may or may not cover a major part of the projection independent space (21), depending on the luck in the abelian projection choice. The maximal abelian projection seems to be the "lucky" one.

Our next problem is to introduce a set of collective coordinates in the projection independent "monopole" space $[\bar{A}]$.

4. Dynamical abelian projection

Though we know that the "monopole" subspace associated with the maximal abelian gauge covers a sufficiently large part of the dominant set $[\bar{A}]$, technical difficulties to find a parametrization of the corresponding subspace $[{}^{\chi}\bar{A}]$ are overwhelming. One has to solve equation (17) for the maximal abelian gauge (19) and a generic configuration A_μ, which is a non-linear differential equation for Ω_χ. If a solution $\Omega_\chi(A)$ yields a non-zero monopole distribution (12), then the corresponding configuration A_μ belongs to the dominant subspace. In addition, one has to bear in mind that equation (17) may have many solutions (for a fixed A_μ) that are Gribov's copies of each other. In the lattice QCD, the Gribov problem can be resolved numerically [13], though it does not seem to be of some relevance [5].

In contrast to the maximal abelian projection, abelian projections based on a local gluon operator Γ in the adjoint representation is technically simpler because the solving of (17) implies a diagonalization of the matrix $\Gamma(A)$. However there is neither theoretical nor numerical proof of the monopole dominance in such abelian projections. They are also often in conflict with the manifest Lorentz invariance.

To resolve the technical difficulties, we propose a new dynamical abelian projection that does not rely on any explicit gauge condition imposed on gauge field configurations. Its lattice version was suggested and studied in [14]. The advantages of this projection are: *(i)* monopole configurations are the only topological defects in this projection, *(ii)* they are parametrized by a real scalar field in the adjoint representation, *(iii)* the projection is Lorentz invariant, and *(iv)* the Gribov problem is avoided. Let us extend the lattice approach of [14] to the continuum case.

Consider the identity

$$1 = \sqrt{\det(-D_\mu^2)} \int \mathcal{D}\phi e^{-\frac{1}{2}\int d^4x \mathrm{tr}(D_\mu\phi)^2} \ , \tag{23}$$

where $D_\mu\phi = \partial_\mu\phi + ig[A_\mu, \phi]$, i.e. the scalar field ϕ realizes the adjoint representation of the gauge group, and substitute it in the path integral (1) for the partition function. As a result we obtain a theory where gauge fields are coupled to an auxiliary scalar field in the adjoint representation. The idea is to use the auxiliary scalar field to make an abelian projection. The effective action is invariant under the gauge transformations (4) and $\phi \to \phi^\Omega = \Omega\phi\Omega^\dagger$. Instead of using a local gluon operator $\Gamma(A)$ to make an abelian projection we set $\Gamma = \Gamma(\phi) = \phi$ and require

$$\Gamma^{off} = \phi^{off} = 0 \ . \tag{24}$$

The gauge condition (24) breaks the gauge group G to its maximal abelian subgroup G_H.

The projection is carried out as follows. Given a configuration ϕ, we find Ω_ϕ such that $\Omega_\phi\phi\Omega_\phi^\dagger = h$ belongs to the Cartan subalgebra. The initial set of field configurations (ϕ, A_μ) is then lifted onto the gauge condition surface (24) by a gauge transformation with the group element Ω_ϕ

$$(\phi, A_\mu) \to (h, A_\mu^\phi = \Omega_\phi A_\mu \Omega_\phi^\dagger + ig^{-1}\Omega_\phi\partial_\mu\Omega_\phi^\dagger) \ . \tag{25}$$

Finally, we split A_μ^ϕ into a sum (8) of its diagonal (Cartan) and off-diagonal components. The residual abelian gauge transformations assume the form (9), while the field h is invariant. If we were able to calculate the integral over h (it is not gaussian due to the Faddeev-Popov determinant associated with the unitary gauge (24)), then we would get a non-local functional of A_μ as a pre-exponential factor in (1) that would be invariant only with respect to the maximal abelian group G_H. Thus, the condition (24) is not a gauge condition imposed directly on gauge fields, but nonetheless it breaks the gauge symmetry to the maximal abelian subgroup through the dynamical coupling of the auxiliary scalar and gauge fields. For this reason we call it the dynamical abelian projection.

Now we turn to the analysis of singular configurations that can occur in the dynamical abelian projection (25). The group element Ω_ϕ is ill-defined at spacetime points where the Jacobian $\mu(h)$ of the change of variables $\phi = \Omega_\phi^\dagger h \Omega_\phi$ vanishes. It is not hard to calculate it [14]. If we denote $d\phi = \mu(h) dh d\mu_H(\Omega_\phi)$, where $d\mu_H(\Omega_\phi)$ is a Haar measure on G/G_H, then for SU(2) and SU(3) we get respectively

$$\mu(h) \;=\; \mu(\phi) = \mathrm{tr}\,\phi^2 \;, \tag{26}$$

$$\mu(h) \;=\; \mu(\phi) = \tfrac{1}{2}(\mathrm{tr}\,\phi^2)^3 - (\mathrm{tr}\,\phi^3)^2 \;. \tag{27}$$

For an arbitrary group, the Jacobian can be found in [14]. In the case of SU(2), the Jacobian vanishes when the scalar field has zeros, $\phi_a(x) = 0$. This imposes three conditions on four spacetime coordinates and therefore determines a set of world lines; the same holds for SU(n) [14]. Thus, Ω_ϕ is ill-defined on some set of worldlines that are formed by zeros of a gauge invariant polynom $\mu(\phi)$. To show that these worldlines are worldlines of magnetic monopoles, one has to repeat the analysis (10)–(16) of section 2, replacing the operator Γ by ϕ.

We shall not do this, instead we give an example of a monopole located at the origin $\mathbf{x}_m = 0$. The field ϕ_a must vanish at $\mathbf{x}_m = 0$, so we choose $\phi_a = x_a f(\mathbf{x})$ (time is fixed), $f(0) = \mathrm{const}$. It is easy to find a matrix $\Omega_\phi \in SU(2)$ that diagonalizes the 2×2 matrix $\phi = \phi_a \tau_a$. We obtain for (14)

$$\bar{C}_\mu^\phi = g_m\,(1 + x_3/r)\,\partial_\mu \tan^{-1}(x_2/x_1)\;, \qquad r = |\mathbf{x}|\;, \qquad g_m = (2g)^{-1} \tag{28}$$

which is the vector potential of the Dirac monopole with the magnetic charge g_m. The curl of the monopole vector potential can be split into a sum of the Coulomb field

$$\bar{\mathbf{B}}_{coul} = g_m \boldsymbol{\nabla} r^{-1}\;, \tag{29}$$

and the string field

$$\bar{\mathbf{B}}_{st} = g_m \boldsymbol{\eta}_3\,(1 + x_3/r)\,[\partial_1, \partial_2]\varphi(x_1, x_2)\;, \tag{30}$$

where $\varphi(x_1, x_2)$ is the polar angle on the x_1x_2-plane and $\boldsymbol{\eta}_3$ is the unit vector pointing in the x_3-direction. Clearly, the Dirac string is located at the semiaxis $x_1 = x_2 = 0$ and $x_3 > 0$. The two terms in the flux integral (15) correspond to contributions of (29) and (30), respectively. The string field (30) is not observable, meaning that it should not contribute to the monopole magnetic field energy. In the next section we give an explicit

construction of configurations that do not give rise to observable Dirac strings upon the dynamical projection.

Thus, all topological singularities occurring upon the dynamical abelian projection are magnetic monopoles; their charges are given by the topological number (12) times the inverse coupling constant $(2g)^{-1}$, and their locations are determined by zeros of the gauge invariant polynom $\mu(\phi)$ (cf. (26) and (27)). The auxiliary scalar field ϕ can be viewed as the "monopole" field that carries all information about monopoles in the dynamical abelian projection.

To make a use of the dynamical abelian projection in the continuum limit, we need to establish a fact of the monopole dominance in this projection. We now proceed to prove it and give an explicit parametrization of the dominant configurations via the auxiliary scalar filed ϕ.

5. Universality of the dynamical abelian projection

As has been pointed out in the previous section, the magnetic field of the Dirac string is not observable. To remove a contribution of the Dirac string to the magnetic field energy, we have to seek for such configurations of gauge fields that the associated field strength $F_{\mu\nu}$ does not exhibit any string-like singularity after the projection. With this purpose, consider a configuration

$$\bar{A}_\mu(\phi) = (4ig)^{-1}[e_\phi, \partial_\mu e_\phi] \ . \tag{31}$$

Its characteristic property is

$$D_\mu(\bar{A}(\phi))e_\phi = 0 \ . \tag{32}$$

Since $[D_\mu, D_\nu] = (ig)^{-1}F_{\mu\nu}$ we conclude that $[F_{\mu\nu}, e_\phi] = 0$ and, hence,

$$F_{\mu\nu} = e_\phi G^\phi_{\mu\nu} \ , \tag{33}$$

where $G^\phi_{\mu\nu}$ is a gauge invariant tensor. A simple calculation yields

$$G^\phi_{\mu\nu} = \tfrac{1}{2}\mathrm{tr}\, e_\phi F_{\mu\nu} == (8ig)^{-1}\mathrm{tr}\ (e_\phi[\partial_\mu e_\phi, \partial_\nu e_\phi]) \ . \tag{34}$$

Upon the projection e_ϕ goes over to τ_3 and, hence, $F_{\mu\nu}$ becomes purely abelian. According to (34) it has no string-like singularities. The gauge potential (31) also turns into a purely abelian one

$$\bar{A}_\mu(\phi) \to \tau_3(C^D_\mu + C^{st}_\mu) \ , \tag{35}$$

where C^D_μ is the monopole potential, and C^{st}_μ has a support on the Dirac string; its curl produces the string magnetic field that cancels with the string field coming from the curl of the Dirac potential. In a particular case when $\phi_a \sim x_a$, $\mathbf{C}^{st}$ is constant along the half-line $\theta = 0$ (here $\cos\theta = x_3/r$), whereas its curl is proportional to $\delta(x_1)\delta(x_2)\theta_H(x_3)$ with θ_H being the Heaviside function.

Thus, the configuration (31) has a desired property: upon the projection the associated abelian field strength does not have the singularity along the Dirac string.

We define a conservative monopole current as

$$j^m_\mu = (8\pi)^{-1}\varepsilon_{\mu\nu\sigma\tau}\partial_\nu G^\phi_{\sigma\tau}\ , \qquad \partial_\mu j^m_\mu = 0\ . \tag{36}$$

The magnetic charge density j^m_0 is then determined by $(\mathbf{\nabla}, \mathbf{B})$ as follows from (34) and (12).

The magnetic field energy of monopoles is still divergent because monopoles are point-like particles. To regularize it, we have to give the Dirac monopole a core as prescribed by (20). It can be achieved, for instance, by the replacement

$$\bar{A}_\mu(\phi) \to \bar{A}_\mu(\phi) + \bar{A}^+_\mu(x)e^+_\phi + \bar{A}^-_\mu(x)e^-_\phi\ , \tag{37}$$

where $e^\pm_\phi = \Omega^\dagger_\phi \tau_\pm \Omega_\phi$, $\tau_\pm = \tau_1 \pm i\tau_2$ and $\bar{A}^\pm_\mu$ are the core functions [12], that is, upon the projection the configuration (37) gives rise to a Wu-Yang monopole with a core [12]. In the conclusion we propose an alternative regularization.

Let us now turn to the question of the dominance of the configurations (37). By construction, the subspace $[\bar{A}(\phi)]$ of all such configurations is one of the "monopole" subspaces $[{}^\chi\bar{A}]$ obtained via an abelian projection in section 3. Positions of Dirac monopoles in the dynamical abelian projection are determined by zeros of the gauge invariant polynom $\mu(\phi)$. Since the auxiliary field ϕ fluctuates, the Jacobian $\mu(\phi)$ fluctuates too, so do the positions and charges of the monopoles. Therefore, letting all configurations of ϕ to occur, which is indeed the case according to the identity (23), we can generate *all* possible distributions of monopoles (12) by making the dynamical abelian projection. Hence, the set of monopole distributions $[q_\phi]$ in the dynamical abelian projection is not less than the projection independent set (22)

$$[\bar{q}] \subseteq [q_\phi]\ . \tag{38}$$

On the other hand, we have established a one-to-one correspondence between monopole distributions from the set (22) and the subspace of the dominant configurations $[\bar{A}]$ modulo regular gauge transformations (see (22)). From this observation and the relation (38) follows that the subspace $[\bar{A}(\phi)]$ covers the subspace of the dominant configurations $[\bar{A}]$

$$[\bar{A}]/[\Omega_0] \subseteq [\bar{A}(\phi)]/[\Omega_0]\ . \tag{39}$$

Therefore any configuration from the dominant subspace $[\bar{A}]$ can be represented in the form (37).

One can understand this from another point of view. By definition every configuration $\bar{A}_\mu$ turns into a Dirac monopole with a core after a certain abelian projection χ. In turn the abelian projection implies a gauge rotation of $\bar{A}_\mu$ with a singular group element Ω_χ specified by χ. The relation (39) follows from two facts. First, it is always possible to find such a configuration of the auxiliary field $\phi(x)$ that the group element Ω_ϕ would lead to the very same monopole distribution (12) as does the Ω_χ, and, second, the configuration (37) turns into the Dirac monopole with a core after the gauge rotation (or projection) with the group element Ω_ϕ.

Thus, the dominant configurations are parametrized by an auxiliary scalar field ϕ in the adjoint representation, $\bar{A}_\mu = \bar{A}_\mu(\phi)$, that is, the field ϕ plays the role of the *field*

collective coordinate for the dominant configurations. The trick of inserting the identity (23) into the path integral (1) can be regarded as a way to obtain a right *gauge* and *Lorentz* invariant measure for the collective coordinate. Note that the dominant configurations are determined up to regular gauge transformations, therefore any parametrization of the dominant subspace should be invariant under regular gauge transformations. The latter does hold for our parametrization because the monopole distribution (12) is determined by zeros of the *gauge invariant* polynom $\mu(\phi)$ and the measure for the collective field coordinate in (23) respects the gauge symmetry too.

6. An effective theory for dominant configurations

To develop an effective theory for the dominant configurations, in the integral

$$\langle W_R \rangle = \langle 1 \rangle^{-1} \int \mathcal{D}A_\mu \mathcal{D}\phi \sqrt{\det(-D_\mu^2)} e^{-S_{YM} - \frac{1}{2}\int d^4x \mathrm{tr}(D_\mu \phi)^2} W_R(A) \ , \tag{40}$$

we perform the change of variables

$$A_\mu = \bar{A}_\mu(\phi) + a_\mu \ , \tag{41}$$

and make the gaussian approximation for quantum fluctuations a_μ around the configurations (37). In the new integration variables (ϕ, a_μ) the measure has the standard form $\mathcal{D}A_\mu \mathcal{D}\phi = \mathcal{D}a_\mu \mathcal{D}\phi$ because the new potentials a_μ are obtained by a shift of A_μ on a function of the independent integration variable ϕ. It should be emphasized that this procedure is no semiclassical approximation because $\bar{A}_\mu(\phi)$ are fully quantum (no classical) configurations of gauge fields parametrized by the quantum field ϕ as prescribed by (37).

To regularize a divergence of the path integral (40) caused by the gauge symmetry, it is convenient to use the background gauge

$$\partial_\mu A_\mu + ig[\bar{A}_\mu(\phi), A_\mu] = 0 \ . \tag{42}$$

So, the corresponding Faddeev-Popov determinant is to be inserted into the path integral measure in (40). Provided the monopole dominance holds in the continuum limit (which is supported by the lattice simulations [4, 5]), the background gauge is safe regarding the Gribov problem because the integral over A_μ in (40) is dominated by the background configurations (37), and, therefore, can be calculated by the perturbation theory over a_μ. Doing the Gaussian integral over a_μ, one gets

$$\langle W_R \rangle \approx \langle 1 \rangle^{-1} \int \mathcal{D}\phi e^{-S_{eff}(\phi, J_0)} W_R(\bar{A}(\phi)) \tag{43}$$

for the Wilson loop expectation value. The effective action S_{eff} consists of two parts: the Yang-Mills action of the background monopole configuration (which is nothing but the dual QED action) and the "trace-logs" of three functional determinants taken on the background configuration, which determine the entropy of the topological (monopole)

defects. The integral (43) describes a quantum theory of the dominant gauge field configurations, that give rise to Dirac monopoles upon the dynamical abelian projection, as an effective dynamics of the auxiliary quantum field ϕ.

The integration over a_μ is nothing but an average over small fluctuations of charged and photon fields in abelian projections. Therefore the dependence of the effective action S_{eff} on the external static source J_0 involves only the Coulomb interaction and possible radiative corrections to it (in higher orders of the perturbation theory). The integral (43) determines an average of the Wilson loop over the "monopole" (dominant) configurations. Note that if one restricts the integration domain by configurations of ϕ for which $\mu(\phi) \neq 0$, then the configuration (31) is a pure gauge (no defects) and the core functions vanish so that the ordinary Faddeev-Popov perturbation theory is recovered.

There is a natural generalization of our construction to an arbitrary unitary group. Let $\lambda^p, \lambda^\alpha$ be respectively basises in the non-Cartan and Cartan subspace of a Lie algebra. The dominant configurations can be parametrized as

$$\bar{A}_\mu(\phi) = (4ig)^{-1}[e_\phi^\alpha, \partial_\mu e_\phi^\alpha] + \bar{A}_\mu^p(x) e_\phi^p \ , \tag{44}$$

where $\bar{A}_\mu^p(x)$ are core functions and $e_\phi^{p,\alpha} = \Omega_\phi^\dagger \lambda^{p,\alpha} \Omega_\phi, \ \Omega_\phi \in G/G_H$. Quark fields can also be included.

7. Conclusion

Based on the results of numerical simulations in lattice gauge theories, we have proposed a gauge invariant formulation of the monopole dominance. Our approach relies on the dynamical abelian projection which is manifestly Lorentz invariant and exempt of the Gribov problem. Assuming that the result of numerical simulations, that the monopole dominance occurs in at least one abelian projection, holds in the continuum limit, we have proved the monopole dominance for the dynamical abelian projection. The latter enables us to obtain an effective theory for the dominant configurations in pure Yang-Mills theory. A further study of the effective theory will be given elsewhere.

As a final remark we mention also that there exists an abelian projection of the theory (40) that interpolates the maximal and dynamical abelian projections:

$$\phi^{off} - i\kappa[\chi_{ma}(A), \phi^H] = 0 \ , \tag{45}$$

where the Lie-algebra-valued function $\chi_{ma}(A)$ is defined by the first equality in (19), $\phi^{off,H}$ are off-diagonal and Cartan components of ϕ, respectively. A constant κ plays the role of the interpolating parameter. The maximal abelian projection is reached in the limit $\kappa \to \infty$, while the dynamical abelian projection corresponds to $\kappa = 0$. Applying arguments similar to those in [2], one can convince oneself that the mass parameter $\kappa^{1/2}$ provides a regularization of topological defects occurring in the κ-projection (45). This could be used as an alternative regularization of the monopole effective action.

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References

[1] G. 't Hooft, in *High Energy Physics*, ed, by A. Zichichi (Editrice Compositori, Bolonga, 1976);
S. Mandelstam, Phys.Rep. 23C (1976) 245.

[2] G. 't Hooft, Nucl.Phys.B 190 (1981) 455.

[3] A.S. Kronfeld, G. Schierholz and U.-J. Wiese, Nucl.Phys.B 293 (1987) 461.

[4] J.D. Stack, S.D. Neiman and R.J. Wensley, Phys.Rev.D 50 (1994) 3399;
H. Shiba and T. Suzuki, Phys.Lett.B 333 (1994) 461

[5] G.S. Bali, V. Bornyakov, M. Mueller-Preussker and K. Schilling, *Dual Scenario of Confinement: A Systemetic Study of Gribov Copy Effects*, hep-lat/9603012;
A. Hart and M. Tepper, *Gribov Copies in the Maximal Abelian Gauge and Confinement*, hep-lat/9606007.

[6] S. Thurner, M. Feurstein, H. Markum and W. Sakuler, Phys. Rev. D 54 (1996) 3457.

[7] A.M. Polyakov, Nucl.Phys.B 120 (1977) 429;
N. Seiberg and E. Witten, Nucl.Phys.B 426 (1994) 19.

[8] L. Del Debbio, A. Di Giacomo, G. Paffuti and P. Pieri, Phys.Lett.B 355 (1995) 255.

[9] I.M. Singer, Commun.Math.Phys. 60 (1978) 7;
M.A. Soloviev, Theor.Math.Phys. (USSR) 78 (1989) 117

[10] M.N. Chernodub, M.I. Polikarpov and V.I. Veselov, Phys.Lett.B 342 (1995) 303.

[11] T. Suzuki and I. Yotsuyanagi, Phys.Rev.D 42 (1990) 4257.

[12] T. Banks, R. Myerson and J. Kogut, Nucl.Phys.B 129 (1977) 493.

[13] S. Hioki, S. Kitahara, Y. Matsubara, O. Migamura, S. Ohno and T. Suzuki, Phys.Lett.B 271 (1991) 201.

[14] S.V. Shabanov, Mod.Phys.Lett.A 11 (1996) 1081;
in *The Proceedings of QCD'96, Montpellier, France*, Nucl.Phys.B (Proc.Suppl.) to appear.